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LETTER TO THE EDITOR

On inverse recursion operator and tri-Hamiltonian formulation for a Kaup–Newell system of DNLS equations

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Abstract. An inverse of the recursion operator is computed and a clear presentation of a tri-Hamiltonian formulation is provided for a Kaup–Newell system of derivative nonlinear Schrödinger (DNLS) equations. Therefore all Kaup–Newell systems in the whole hierarchy of DNLS equations are tri-Hamiltonian and have an inverse hierarchy of common commuting symmetries.

The Kaup–Newell system of derivative nonlinear Schrödinger (DNLS) equations:

$$u_t = \begin{pmatrix} q_t \\ r_t \end{pmatrix} = K(u) = \frac{1}{2}\begin{pmatrix} q_{xx} - (q^2r)_x \\ -r_{xx} - (qr^2)_x \end{pmatrix} \tag{1}$$

is a system of typical soliton equations. This system originates from an investigation on a derivative nonlinear Schrödinger equation and the massive Thirring model by Kaup and Newell [1, 2]. It is found that it has a Lax pair [3]

$$U = \begin{pmatrix} \lambda & q \\ \lambda r & -\lambda \end{pmatrix} \qquad V = \begin{pmatrix} \lambda^2 - \frac{1}{2}\lambda qr & \lambda q + \frac{1}{2}(q_x - q^2 r) \\ \lambda^2 r - \frac{1}{2}\lambda(r_x + qr^2) & -\lambda^2 + \frac{1}{2}\lambda qr \end{pmatrix}$$

with λ being a spectral parameter. The spectral matrix operator U is different from the original one in [1,2] but it can still generate the whole Kaup–Newell hierarchy of DNLS equations [3]. It follows that the Kaup–Newell system (1) may equivalently be derived from the zero-curvature equation

$$U_t - V_x + [U, V] = 0$$

under the isospectral condition $\lambda_t = 0$, and it is already shown that the Kaup–Newell system (1) possesses a hereditary recursion operator (see, for example, [3])

$$\Phi = \begin{pmatrix} \frac{1}{2}\partial - \frac{1}{2}\partial q\partial^{-1}r & -\frac{1}{2}\partial q\partial^{-1}q \\ -\frac{1}{2}\partial r\partial^{-1}r & -\frac{1}{2}\partial - \frac{1}{2}\partial r\partial^{-1}q \end{pmatrix} \qquad \partial\partial^{-1} = \partial^{-1}\partial = 1 \qquad \partial = \frac{\partial}{\partial x}. \tag{2}$$

In this letter, we want to give an explicit inverse of the recursion operator Φ defined by (2) and to provide a clear presentation of a tri-Hamiltonian formulation for the Kaup–Newell system of DNLS equations (1). Therefore, the Kaup–Newell system (1) will provide an example of tri-Hamiltonian systems, among which the well known examples are the coupled KdV systems [4], the Toda lattice [5] and the Volterra lattice [6].

Let us start from a set of Hamiltonian operators

$$J(\alpha)=\begin{pmatrix} -2q\partial^{-1}q & 2+\alpha\partial+2q\partial^{-1}r \\ -2+\alpha\partial+2r\partial^{-1}q & -2r\partial^{-1}r \end{pmatrix} \tag{3}$$

where α can be any constant. These operators are a simple generalization of the AKNS case [7]. The proof that $J(\alpha)$ is Hamiltonian is a rather laborious computation but it is direct and similar to the AKNS case. A special choice leads to a Hamiltonian pair

$$J_0=\begin{pmatrix} -2q\partial^{-1}q & 2+2q\partial^{-1}r \\ -2+2r\partial^{-1}q & -2r\partial^{-1}r \end{pmatrix} \qquad J_1=\begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}. \tag{4}$$

Importantly, by inspection, we can neatly present the inverse operator of J_0:

$$J_0^{-1}=-\frac{1}{2}\begin{pmatrix} r\partial^{-1}r & 1+r\partial^{-1}q \\ -1+q\partial^{-1}r & q\partial^{-1}q \end{pmatrix}$$

which can now be shown by checking that $J_0J_0^{-1}=J_0^{-1}J_0=I_2$ (I_2 being the 2×2 identity matrix). Therefore the above Hamiltonian pair engenders [8] a hereditary symmetry operator $\Phi=J_1J_0^{-1}$ which is exactly the same as the hereditary recursion operator defined by (2). Now the inverse of the recursion operator Φ, which is still a hereditary recursion operator for the Kaup–Newell system (1) (see [9], for example), is determined by

$$\Phi^{-1}=J_0J_1^{-1}=\begin{pmatrix} 2\partial^{-1}+2q\partial^{-1}r\partial^{-1} & -2q\partial^{-1}q\partial^{-1} \\ -2r\partial^{-1}r\partial^{-1} & -2\partial^{-1}+2r\partial^{-1}q\partial^{-1} \end{pmatrix}. \tag{5}$$

This provides us with an explicit and nice expression for the inverse of the recursion operator Φ. Such nice expressions of inverses of recursion operators have not been found in the cases of the AKNS hierarchy and the Jaulent–Miodek hierarchy, for which special eigenfunctions of the related spectral problems are involved in the construction of inverses of recursion operators [10, 11]. A general theory on Hamiltonian operators and Hamiltonian pairs can be found in [12–14]. Within our discussion, we just pick out a set of specific Hamiltonian operators from (3) for our purpose. Actually, other types of hereditary operators may also be constructed [15].

Let us now assume that

$$J_2=\Phi J_1=\begin{pmatrix} -\frac{1}{2}\partial q\partial^{-1}q\partial & \frac{1}{2}\partial^2-\frac{1}{2}\partial q\partial^{-1}r\partial \\ -\frac{1}{2}\partial^2-\frac{1}{2}\partial r\partial^{-1}q\partial & -\frac{1}{2}\partial r\partial^{-1}r\partial \end{pmatrix} \tag{6}$$

and then three operators J_0, $J_1=\Phi J_0$ and $J_2=\Phi^2J_0$ constitute a Hamiltonian triple, which means that any linear combination of J_0, $J_1=\Phi J_0$ and $J_2=\Phi J_1$ is again Hamiltonian. This is a consequence of a Hamiltonian pair with an invertible Hamiltonian operator.

Now we are ready to give a clear presentation of a tri-Hamiltonian formulation for the Kaup–Newell system of DNLS equations (1). The tri-Hamiltonian formulation that we are looking for reads as

$$u_t=K(u)=J_0\frac{\delta\tilde{H}_2}{\delta u}=J_1\frac{\delta\tilde{H}_1}{\delta u}=J_2\frac{\delta\tilde{H}_0}{\delta u} \tag{7}$$

where three Hamiltonian functionals are given by $\tilde{H}_i=\int H_i\,\mathrm{d}x$, $0\leqslant i\leqslant 2$, with

$$\begin{cases} H_0=qr \qquad H_1=-\frac{1}{4}q^2r^2-\frac{1}{4}qr_x+\frac{1}{4}q_xr \\ H_2=\frac{1}{16}q_{xx}r+\frac{1}{16}qr_{xx}-\frac{1}{8}q_xr_x+\frac{3}{16}q^2rr_x-\frac{3}{16}qq_xr^2+\frac{1}{8}q^3r^3 \end{cases} \tag{8}$$

and as usual, the variational derivative of a functional $\tilde{H}=\int H\,\mathrm{d}x$ is defined by

$$\frac{\delta\tilde{H}}{\delta u}=\left(\frac{\delta\tilde{H}}{\delta q},\frac{\delta\tilde{H}}{\delta r}\right)^T \qquad \frac{\delta\tilde{H}}{\delta q}=\sum_{i\geqslant 0}(-\partial)^i\frac{\partial H}{\partial q^{(i)}}$$

$$\frac{\delta\tilde{H}}{\delta r}=\sum_{i\geqslant 0}(-\partial)^i\frac{\partial H}{\partial r^{(i)}}\qquad q^{(i)}=\frac{\partial^i q}{\partial x^i}\qquad r^{(i)}=\frac{\partial^i r}{\partial x^i}.$$

The above functions H_0, H_1 and H_2 are all conserved densities of the Kaup–Newell system (1). The proof of the tri-Hamiltonian formulation (7) just needs a direct computation. The second Hamiltonian formulation is easy to discern, because a special constant coefficient and skew-symmetric differential operator is taken as the Hamiltonian operator. The third one is a direct consequence of the recursion structure of the Kaup–Newell hierarchy. However the first Hamiltonian structure is not so obvious. The success in getting it here is a decomposition $J_1 = \Phi J_0$ designed for the simple Hamiltonian operator J_1. The above three Hamiltonian operators J_0, J_1 and J_2 go from complicated through simple to complicated structures. This is different from the normal case from simple to complicated structures (see, for example, [4,5]). We point out that parts of the tri-Hamiltonian formulation (7) were also discussed from different points of view in [16–19], but there was neither a clear presentation of the whole tri-Hamiltonian formulation (7), nor the proof or statement on the compatibility of all three operators J_0, J_1 and J_2, i.e., the Hamiltonian property of $c_0J_0+c_1J_1+c_2J_2$ with arbitrary constants c_0, c_1 and c_2, for presenting the tri-Hamiltonian formulation of (1).

Based on the Magri's scheme of bi-Hamiltonian formulation [12], it follows from (7) that each nonlinear Kaup–Newell system of DNLS equations among the hierarchy

$$u_t=K_n=\Phi^n u_x\qquad n\geqslant 0 \tag{9}$$

has a tri-Hamiltonian formulation

$$u_t=K_n=J_0\frac{\delta\tilde{H}_{n-1}}{\delta u}=J_1\frac{\delta\tilde{H}_n}{\delta u}=J_2\frac{\delta\tilde{H}_{n+1}}{\delta u}\qquad n\geqslant 1 \tag{10}$$

the first nonlinear system of which is exactly the Kaup–Newell system (1). The existence of all Hamiltonian functionals $\tilde{H}_n$ is guaranteed by a specific pair of Hamiltonian operators J_0 and J_1 defined by (4). In fact, they can be computed as follows:

$$\tilde{H}_n=\int H_n\,\mathrm{d}x\qquad H_n=\int_0^1 u^T G_n(\lambda u)\,\mathrm{d}\lambda\qquad G_n(u)=\Psi^n\begin{pmatrix} r\\ q\end{pmatrix} n\geqslant 0 \tag{11}$$

where T means the transpose of matrices and Ψ is the conjugate operator $\Phi^\dagger$ of Φ.

Since Φ and Φ^{-1} are all hereditary, it is easy to obtain

$$[K_m,K_n]:=\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0}(K_m(u+\varepsilon K_n)-K_n(u+\varepsilon K_m))=0\qquad K_n=\Phi^n u_x,\quad m,n\in\mathbb{Z}.$$

Therefore, through the inverse operator Φ^{-1} given by (5), an inverse hierarchy of common commuting symmetries can be computed as follows:

$$u_t=K_{-n}=\Phi^{-n}u_x\qquad n\geqslant 1 \tag{12}$$

the first and the second of which are

$$\begin{pmatrix} q_t\\ r_t\end{pmatrix}=\begin{pmatrix} 2q\\ -2r\end{pmatrix}\qquad \begin{pmatrix} q_t\\ r_t\end{pmatrix}=\begin{pmatrix} 4\partial^{-1}q+4q\partial^{-1}r\partial^{-1}q+4q\partial^{-1}q\partial^{-1}r\\ 4\partial^{-1}r-4r\partial^{-1}r\partial^{-1}q-4r\partial^{-1}q\partial^{-1}r\end{pmatrix}.$$

The second symmetry system is nonlocal and so are the other nonlinear symmetries in the inverse hierarchy (12). Nevertheless, the explicit expression (5) of the inverse of the hereditary recursion operator Φ brings us a great convenience to present the inverse hierarchy [20] of commuting symmetries.

Finally, we remark that a Hamiltonian pair with an invertible operator can provide a quadruple (and even more multiple) of Hamiltonian operators [8]. For example, in our case, we can have a quadruple of Hamiltonian operators J_0, ΦJ_0, $\Phi^2 J_0$ and $\Phi^3 J_0$, which means that all operators $c_0 J_0 + c_1 \Phi J_0 + c_2 \Phi^2 J_0 + c_3 \Phi^3 J_0$ with arbitrary constants c_0, c_1, c_2 and c_3 are Hamiltonian. However, we conjecture that there is no quadruple Hamiltonian formulation with four Hamiltonian operators J_0, ΦJ_0, $\Phi^2 J_0$ and $\Phi^3 J_0$ and four local Hamiltonian functionals for the Kaup–Newell system of DNLS equations (1), but we have no idea whether there exist other quadruple Hamiltonian formulations with local Hamiltonian functionals for the Kaup–Newell system of DNLS equations (1).

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